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New Notion of Open Sets in Penta Topological Spaces

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Abstract

In this paper, we introduce the notion of Penta topological space and investigate the fundamental concepts in classical topological spaces for Penta topological spaces. To do so we define new types of open sets and closed sets namely, Penta semi-alpha open set in the setting of Penta topological space. We also study the idea of Penta semi-alpha-continuity in Penta topological spaces.

Keywords: Penta topological space, p-open set, p-closed set, p-continuity, $(s\alpha)_p$-open set, $(s\alpha)_p$-closed set, $(s\alpha)_p$-continuity.

I. Introduction

In recent years the concept of a single topological space has been extended to bi-topological space (a non-vacuous set X endowed with two topologies $\tau 1$ and $\tau 2$), tri-topological space (a non-vacuous set X endowed with three topologies $\tau 1$, $\tau 2$ and $\tau 3$) and quad-topological space (a non-vacuous set X endowed with four topologies $\tau 1$, $\tau 2$, $\tau 3$ and $\tau 4$). The concept of a bi-topological space was first introduced by Kelly [1]. Tri-topological space was initiated by Kovar [2]. Quad-topological space was investigated by Mukundan [4]. Tapi and Sharma [5] studied the idea of Q-B continuous functions in quad topological spaces. As a natural generalization of these concepts, Muhammad Shahkar Khan and Gulzar Ali Khan[3] introduce a new concept called penta-topological space. A penta-topological space (X, τ) is a set X equipped with 5-tuple of topologies τ = ($\tau 1$, $\tau 2$, $\tau 3$, $\tau 4$, $\tau 5$) - called penta topology on X. In this paper we introduce the concept of topological structures with penta topology and define new types of open (closed) sets namely, Penta semi alpha open set in penta topological spaces. We also introduce the notion of $(s\alpha)_p$-continuous function in penta- topological spaces.

II. Preliminaries

Definition 2.1

Let (X, τ_p) be a penta-topological space. Elements of τ_P; $P \in \{1,2,3,4,5\}$ are called τ_P-open sets and their relative complements are called τ_P-closed sets.

Definition 2.2

Let (X, τ_p) be a penta -topological space. A subset A of X is called penta-open (p-open) if $A \in\cup\ \tau_P$, $P \in \{1, 2, 3, 4, 5\}$ and its complement is said to be penta-closed (p-closed).

The collection of all penta closed sets denoted as $C_P(A)$.

The collection of all penta open sets denoted as $O_P(A)$.

Definition 2.3

Let (X, τ_p) be a penta-topological space. Let A be a subset of X. The penta-closure of A, denoted by $cl_p(A)$ is defined as the intersection of all penta-closed sets of X containing A. Thus, if $\{C_\alpha : \alpha \in I\}$ is the collection of all penta-closed sets in X containing A, then $cl_p(A) = \cap\, \alpha \epsilon I\ C_\alpha$

Remarks 2.4

1) $A \subseteq cl_p(A)$. Since intersection of penta-closed sets is penta-closed, $cl_p(A)$ is a penta- closed set.
2) $cl_p(A)$ is the smallest penta-closed set containing A.
3) $int_p(A)$ is the union of all p-open sets contained in A and hence the largest penta-open set contained in A.
4) φ and X are both penta open and penta closed.
5) Arbitrary union of penta open sets is penta open.
6) Arbitrary intersection of penta closed sets is penta closed.
7) Penta open sets satisfy all the axioms of topology.

Definition 2.5

Let (X, τ_p) be a penta topological space. $A\subseteq$ is called

1) penta semi open(briefly- $S_P O$s) if $A\subseteq cl_p\big(int_p(A)\big)$.
2) Penta pre open(briefly- $P_P O$s) if $A\subseteq int_p\big(cl_p(A)\big)$.
3) Penta alpha open(briefly- $\alpha_P O$s) if $A \subseteq int_p\big(cl_p(int_p A)\big)$

III. Penta Semi Alpha Open Sets

Definition 3.1 Let (X, τ_p) be a penta topological space then a subset A of X is said to be penta semi alpha open set(briefly $s\alpha_P$-Os) if $A \subseteq cl_P\big(\alpha_P int(A)\big)$ and complement of penta semi alpha open set is penta semi alpha closed set.

Proposition 3.2 It is evident by defintions that in a penta topological space (X, τ_p) the following are hold :

a) Every O_P(resp.C_P)- sets is a $s\alpha_P$-Os(resp. $s\alpha_P$-Cs)
b) Every $\alpha_P O$(resp. $\alpha_P C$)- sets is a $s\alpha_P$-Os(resp. $s\alpha_P$-Cs)

Remark 3.3 Converse of the above proposition need not be true as seen from the following example:

Example 3.4 Let $X = \{a, b, c, d\}$. Consider the topologies

$\tau_1 = \{\phi, \{a,d\}, X\}$, $\tau_2 = \{\phi, \{d\}, X\}$, $\tau_3 = \{\phi, \{a,b\}, X\}$, $\tau_4 = \{\phi, \{a,b,d\}, X\}$, $\tau_5 = \{\phi, X\}$ be a penta topological space.

Then O_P- set: $\{\phi, X, \{d\}, \{a,b\}, \{a,d\}, \{a,b,d\}\}$

$S\alpha_P O$-sets : $\{\phi, X, \{d\}, \{a,b\}, \{a,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}\}$

Let $A = \{a,c,d\}$ is $\alpha_P O$-set but not O_P-set.Also $B = \{a,b,c\}$ is $s\alpha_P$O-set but not $\alpha_P O$-set.

Remark 3.5 The concepts of $S\alpha_P$-O sets and $P_P O$-set are independent as the following examples shows:

Example 3.6 From example 3.4, Let $C = \{a\}$ is $P_P O$-set but not $S\alpha_P$-O set.

Example 3.7 Let $X = \{a,b,c,d\}$. Consider the topologies

$\tau_1 = \{\phi, \{a\}, X\}$, $\tau_2 = \{\phi, \{d\}, \{a,d\}, X\}$, $\tau_3 = \{\phi, \{c\}, X\}$, $\tau_4 = \{\phi, \{c,d\}, X\}$,

$\tau_5 = \{\phi, X, \{a,c\}, \{a,c,d\}\}$ be a penta topological space.

Let $D = \{a,b,d\}$ is $S\alpha_P$-O set but not $P_P O$-set.

Proposition 3.7 Let A & B be subsets of (X, τ_p) such that $B \subseteq A \subseteq cl_P(B)$. If B is $S\alpha_P$-O set then A is also $S\alpha_P$-O set.

Proof: Given B is $S\alpha_P$-O set. So we have $B \subseteq cl_P(\alpha_P int(B)) \subseteq cl_P(\alpha_P int(A))$. Thus $B \subseteq cl_P(\alpha_P int(A))$. Hence A is $S\alpha_P$-O set.

Proposition 3.8 For any penta subset A of a penta topological space (X, τ_p) , $A \subseteq \alpha_P O(X)$ if and only if there exists a O_P-set B such that $B \subseteq A \subseteq int_P(Cl_P(B))$.

Proof : Let A be a $\alpha_P O(X)$. Hence $A \subseteq int_P\left(Cl_P(int_P(A))\right)$. So let $B = int_P(A)$, we get $int_P(A) \subseteq A \subseteq int_P\left(Cl_P(int_P(A))\right)$. Then there exists a O_P-set $int_P(A)$ such that $B \subseteq A \subseteq int_P(Cl_P(B))$, where $B = int_P(A)$.

Conversely suppose that there is a O_P-set B such that that $B \subseteq A \subseteq int_P(Cl_P(B))$.

To prove: $A \in \alpha_P O(X)$

$B \subseteq int_P(A)$ [since $int_P(A)$ is the largest O_P-set contained in A]

Hence $Cl_P(B) \subseteq Cl_P(int_P(A))$, then $int_P(Cl_P(B)) \subseteq int_P\left(Cl_P(int_P(A))\right)$

But $B \subseteq A \subseteq int_P(Cl_P(B))$ [By hypothesis]

Then $A \subseteq int_P\left(Cl_P(int_P(A))\right)$. Therefore $A \in \alpha_P O(X)$.

Proposition 3.9 For any penta subset A of a penta topological space (X, τ_p) the following properties are equivalent:

I. $A \in S\alpha_P$-O(X)
II. There exists O_P-set say B such that $B \subseteq A \subseteq int_P\left(Cl_P(int_P(A))\right)$

III. $A \subseteq Cl_P\left(int_P\left(Cl_P\left(int_P(A)\right)\right)\right)$

Proof : (I)⇒(II) Let $A \in S\alpha_P$-O(X). Then there exists $V \in \alpha_P$-O(X) such that $V \subseteq A \subseteq Cl_P(V)$. Hence there exists a O_P-set B, such that $B \subseteq V \subseteq int_P\left(Cl_P(V)\right)$[by Prev.Prop]. Therefore $Cl_P(B) \subseteq Cl_P(V) \subseteq Cl_P\left(int_P\left(Cl_P(B)\right)\right)$, implies that $Cl_P(V) \subseteq Cl_P\left(int_P\left(Cl_P(B)\right)\right)$. Then $B \subseteq V \subseteq A \subseteq Cl_P(V) \subseteq Cl_P\left(int_P\left(Cl_P(B)\right)\right)$. Therefore, $B \subseteq A \subseteq Cl_P\left(int_P\left(Cl_P(B)\right)\right)$, for some B O_P-set.

(II)⇒(III) Suppose that there exists a O_P-set B such that $B \subseteq A \subseteq Cl_P\left(int_P\left(Cl_P(B)\right)\right)$. We know that, $int_P(A) \subseteq A$. On the other hand, $B \subseteq int_P(A)$[Since $int_P(A)$ is the largest O_P-set contained in A].Hence $Cl_P(B) \subseteq Cl_P\left(int_P(A)\right)$, then $int_P\left(Cl_P(B)\right) \subseteq int_P\left(Cl_P\left(int_P(A)\right)\right)$, therefore $Cl_P\left(int_P\left(Cl_P(B)\right)\right) \subseteq Cl_P\left(int_P\left(Cl_P\left(int_P(A)\right)\right)\right)$. But $A \subseteq Cl_P\left(int_P\left(Cl_P(B)\right)\right)$[By hypothesis]. Hence $A \subseteq Cl_P\left(int_P\left(Cl_P\left(int_P(A)\right)\right)\right)$

(III)⇒(I) Let $A \subseteq Cl_P\left(int_P\left(Cl_P\left(int_P(A)\right)\right)\right)$

To prove : $A \in S\alpha_P$-O(X). Let K=$int_P(A)$. We know that $int_P(A) \subseteq A$

(i.e)To prove: $A \subseteq Cl_P\left(int_P(A)\right)$

Since $int_P\left(Cl_P\left(int_P(A)\right)\right) \subseteq Cl_P\left(int_P(A)\right)$.

Hence $Cl_P\left(int_P\left(Cl_P\left(int_P(A)\right)\right)\right) \subseteq int_P\left(Cl_P\left(int_P(A)\right)\right)$=$Cl_P\left(int_P(A)\right)$

But $A \subseteq Cl_P\left(int_P\left(Cl_P\left(int_P(A)\right)\right)\right)$ [By hypothesis]

Hence, $A \subseteq Cl_P\left(int_P\left(Cl_P\left(int_P(A)\right)\right)\right) \subseteq Cl_P\left(int_P(A)\right)$ imlies that $A \subseteq Cl_P\left(int_P(A)\right)$

Hence there exists a O_P-set say K, such that $K \subseteq A \subseteq Cl_P(A)$.

On the other hand, K is a α_P-Os[Since K is O_P-set. Hence $A \in S\alpha_P$-O(X).

Proposition 3.10 The union of any family of α_P-Os is again a α_P-Os.

Proof: Let $\{A_i\}_{i\in J}$ be a family of α_P-Os of (X, τ_P).

To prove: $\bigcup_{i\in J} A_i$ be a family of α_P-Os of (X, τ_P)

(i.e) $\bigcup_{i\in J} A_i \subseteq int_P\left(Cl_P\left(int_P(A_i)\right)\right)$

Now , $A_i \subseteq int_P\left(Cl_P\left(int_P(A_i)\right)\right)$, $\forall i \in J$

Since $\bigcup_{i\in J} int_P(A_i) \subseteq int_P\left(\bigcup_{i\in J} A_i\right)$ and $\bigcup_{i\in J} Cl_P(A_i) \subseteq Cl_P\left(\bigcup_{i\in J} A_i\right)$ hold for any penta topology.

We have, $\bigcup_{i\in J} A_i \subseteq \bigcup_{i\in J} int_P\left(Cl_P\big(int_P(A_i)\big)\right) \subseteq int_P\left(\bigcup_{i\in J}\left(Cl_P\big(int_P(A_i)\big)\right)\right)$

$\subseteq int_P\left(Cl_P\big(\bigcup_{i\in J}\big(int_P(A_i)\big)\big)\right) \subseteq int_P\left(Cl_P\left(int_P\big(\bigcup_{i\in J} A_i\big)\right)\right)$

Hence $\bigcup_{i\in J} A_i$ is a α_P-Os of (X,τ_P).

Proposition 3.11 The union of any family of $S\alpha_P$-Os is again a $S\alpha_P$-Os.

Proof: Let $\{A_i\}_{i\in J}$ be a family of $S\alpha_P$-Os of (X,τ_P).

To prove: $\bigcup_{i\in J} A_i$ be a family of $S\alpha_P$-Os of (X,τ_P)

(i.e) $\bigcup_{i\in J} A_i \subseteq \left(Cl_P\big(\alpha_P int_P(A_i)\big)\right)$

Since $A_i \in S\alpha_P$-O(X), then there is a α_P-Os B_i such that $B_i \subseteq A_i \subseteq Cl_P(B_i)$, $\forall i \in J$.

Hence $\bigcup_{i\in J} B_i \subseteq \bigcup_{i\in J} A_i \subseteq \bigcup_{i\in J} Cl_P(B_i) \subseteq Cl_P\big(\bigcup_{i\in J} B_i\big)$

But $\bigcup_{i\in J} B_i \in \alpha_P$-O$(X)$[By Prev.Prop]

Hence $\bigcup_{i\in J} A_i \in S\alpha_P$-O$(X)$

Remark 3.12 The intersection of any two $S\alpha_P$-O sets may not be a $S\alpha_P$-O set as show in the following example:

Example 3.13 Let $X = \{a,b,c,d\}$. Consider the topologies

$\tau_1 = \{\emptyset,\{a\},\{b\},\{a,b\},X\}$, $\tau_2 = \{\emptyset,\{a\},\{a,c\},X\}$, $\tau_3 = \{\emptyset,\{a,b,c\},X\}$,

$\tau_4 = \{\emptyset,\{b\},\{c\},\{b,c\},X\}$, $\tau_5 = \{\emptyset,\{c\},\{a,b,c\},X\}$ be a penta topological space.

Here $A = \{b,d\}$ and $B = \{c,d\}$ are $S\alpha_P$-O(X) but $A\cap B = \{d\}$ is not $S\alpha_P$-O(X).

IV. Penta Semi Alpha Continuous Function

Definition 4.1 Let (X,τ_P) and (Y,σ_P) be two penta topological spaces. A function $f: X \to Y$ is called $S\alpha_P$-continuous function if $f^{-1}(V)$ is $S\alpha_P$-open set in (X,τ_P) for every O_P-set V in (Y,σ_P)

Example 4.2 Let $X = Y = \{a,b,c\}$. Consider the topologies

$\tau_1 = \{\emptyset,\{a\},X\}$, $\tau_2 = \{\emptyset,\{b\},X\}$, $\tau_3 = \{\emptyset,\{b,c\},X\}$, $\tau_4 = \{\emptyset,\{a,b\},X\}$, $\tau_5 = \{\emptyset,X\}$ and

$\sigma_1 = \{\emptyset,\{a\},X\}$, $\sigma_2 = \{\emptyset,\{a\},\{a,b\},X\}$, $\tau_3 = \{\emptyset,\{a,c\},X\}$, $\tau_4 = \{\emptyset,\{a,b\},X\}$, $\tau_5 = \{\emptyset,\{b,c\},X\}$ be penta topological space.

Define $f: X \to Y$ be an identity function by $f(a)=\{a\}$, $f(b)=\{b\}$, $f(c)=\{c\}$

The $S\alpha_P$-open set in (X,τ_P) are $\{\emptyset,X,\{a\},\{b\},\{a,b\},\{b,c\}\}$

The O_P-set in (Y,σ_P) are $\{\emptyset,X,\{a\},\{a,b\},\{a,c\},\{b,c\}\}$. Since $f^{-1}(V)$ is $S\alpha_P$-open set in (X,τ_P) for every O_P-set V in (Y,σ_P), hence f is $S\alpha_P$-continuous function.

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